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FOURTH ORDER SCATTERING MATRIX ELEMENTS OF NUCLEONS WITH A FOURTH ORDER MESON EQUATION

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ABSTRACT. In this paper the important fourth order matrix elements with a fourth order meson equation have been calculated. It is noted that the charge renormalisation that we obtain with this theory is finite and small for a reasonable value of the coupling constant. But the contribution from the meson self-energy diagram remains unaltered and thus contains an infinity. A consistent interpretation of this term as meson mass renormalisation is obtained only if the bare meson has two different rest-masses.

INTRODUCTION

In two previous papers, we have considered a fourth order meson equation proposed by Bhabha (1950) and Thirring (1950) It has been noted that this theory is fairly successful for the explanation of the anomalous magnetic moments of nucleons (Misra and Deo, 1956), but fails to give experimental results for neutronproton scattering (Misra, 1960). However, the principal advantage of this theory is the finiteness of many matrix elements that are divergent in conventional meson theory. For example, Thirring noted that the self-energy of the nucleon in this theory was finite and fairly small for a reasonable value of the coupling constant These are well illustrated when we consider the fourth order matrix elements for nucleon-nucleon scattering. It is the purpose of the present paper to calculate these matrix elements

It is noted here that the charge renormalisation that we obtain in this theory from the vertex diagram is finite and small. But, the contribution from the meson self-energy diagram remains unaltered, and thus contains an infinity This happens since the nucleon propagator, on which this depends, remains the same here. The interpretation of this term as mass renormalisation, however, presents serious difficulty, and this infinite renormalisation has been carried out in the appendix. We find that this mass renormalisation can be consistently done to give rise to unique renormalised mass only if the bare mesons had two different rest-masses. On the other hand, a single mass of the bare meson would give rise to two different renormalised masses. This aspect of the problem would be interesting if the renormalisation terms were finite, which unfortunately is not the case here.

We now proceed to evaluate the fourth order matrix elements. The notation here will be the same as in the previous paper (Misra, 1960). We only note that the nucleon propagator is given as

$$<P(\psi(x)\bar{\psi}(y))>_0 = \frac{i}{(2\pi)^4}\int \frac{i\gamma k+\kappa_0}{(k^2+\kappa_0^2)} \exp(ik(x-y))d^4k \tag{1}$$

and the meson propagator, as

$$<P(\phi^i(x)\phi^j(y))>_0 = -\frac{i\kappa^2}{(2\pi)^4}\delta_{ij}\int \frac{\exp(ik(x-y))}{(k^2+\kappa^2)^2} d^4k \tag{2}$$

FOURTH ORDER S-MATRIX ELEMENTS

Here we have,

$$<S_4> = \frac{f^4}{4!}\int d^4x_1 \ldots \int d^4x_4 <P\bar{\psi}(x_1)\gamma_5\tau_i\psi(x_1)\phi^i(x_1)\ldots$$

$$\bar{\psi}(x_4)\gamma_5\tau_l\psi(x_4)\phi^l(x_4))> \quad \ldots \quad (3)$$

where the expectation value is to be taken between initial and final two-nucleon states with four momenta p_1, p_2 and p_3, p_4 respectively. This gives rise to different Feyman digrams which we consider separately.

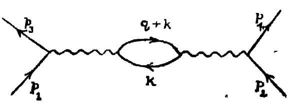

Fig. 1(a)

(*a*) *Vacuum polarisation* :

Here we obtain

$$<S_4> = \frac{f^4\kappa^1}{4!}\cdot {}^4C_2\cdot 2\cdot\frac{\kappa_0^2}{(2\pi)^6(p_{10}p_{20}p_{30}p_{40})^{\frac{1}{2}}}\quad\delta(p_1+p_2-p_3-p_4)$$

$$\times\bar{u}(p_3)\gamma_5\tau_i u(p_1)\bar{u}(p_4)\gamma_5\tau_j u(p_2)\times\int d^4k\{Sp[\gamma_5\tau_i(i\gamma(q+k)-\kappa_0)\gamma_5\tau_j(i\gamma k-\kappa_0)]$$
$$(q^2+\kappa^2)^{-4}((q+k)^2+\kappa_0^2)^{-1}(k^2+\kappa_0^2)^{-1}\}.$$

which simplifies to

$$S_4^{(a)} = \frac{4f^4\kappa^4\kappa_0^2}{(2\pi)^6(p_{10}p_{20}p_{30}p_{40})^{\frac{1}{2}}}\delta(p_1+p_2-p_3-p_4)(q^2+\kappa^2)^{-4}\, I$$

$$\times\bar{u}(p_3)\gamma_5\tau_i\psi(p_1)\bar{\psi}(p_4)\gamma_5\tau_i\psi(p_2) \quad\ldots\quad (4)$$

where

$$I = \int d^4k \frac{(k^2+qk+\kappa_0^2)}{(k^2+2qk+q^2+\kappa_0^2)(k^2+\kappa_0^2)}$$

The above integral is divergent, and we can write, as has been shown in the appendix.

$$I = I(q^2) = A+(q^2+\kappa^2)B+(q^2+\kappa^2)^2 I_c(q^2) \qquad \ldots \quad (6)$$

where A and B are respectively quadratically and logarithmically divergent constants, and $I_c(q^2)$ gives rise to a finite contribution. We shall also show in the appendix that both A and B go as mass renormalisation terms. We note that the same constants also occur in conventional meson theory; but there the interpretation is different, A alone going as mass renormalisation, and B giving rise to coupling constant renormalisation (Schweber, Bethe and Hoffman, 1955).

After renormalisation, the (finite) contribution from this diagram has been evaluated in the appendix as

$$(S^{(a)}{}_4)_{rn} = \frac{4f^4\kappa_0^2\kappa^4}{(2\pi)^6(p_{10}p_{20}p_{30}p_{40})^{1/2}} \delta(p_1+p_2-p_3-p_4)$$

$$\times \bar{u}(p_3)\gamma_5\tau_i u(p_1) \quad \bar{u}(p_4)\gamma_5\tau_i u(p_2)(q^2+\kappa^2)^{-2}$$

$$\times i\pi^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \;\; x^2(1-x)^2y^2\{3(\kappa_0^2+q^2x(1-x)yz$$

$$-\kappa^2x(1-x)(1-yz))^{-1}+2\kappa_0^2(\kappa_0^2+q^2x(1-x)yz-\kappa^2x(1-x)(1-yz)^{-1}\} \quad \ldots \quad (7)$$

In the nonrelativistic approximation, neglecting terms of the order of (q^2/κ_0^2) and $(\kappa^2/\kappa_0^2$ as compared to unity, we get

$$(S^{(a)}{}_4)_{rn} = \frac{2i\pi^2f^4\kappa^4}{9\kappa_0^2(2\pi)^6} \quad \delta(p_1+p_2-p_3-p_4)(q^2+\kappa^2)^{-2}\times$$

$$\times \bar{u}(p_3)\gamma_5\tau_i u(p_1)\bar{u}(p_4)\gamma_5\tau_i u(p_2).$$

Thus we find that this contribution, which is of the same type as the second order matrix element, is of the order of$(f\kappa/\kappa_0)^2$ times the second order matrix element.

(*b*) *Vertex diagram* :

We see that the contribution from the vertex diagram (Fig. 1b) is

$$S^{(b)}_4 = \frac{f^4\kappa^4\kappa_0^2}{(2\pi)^6(p_{10}p_{20}p_{30}p_{40})^{\frac{1}{2}}}\,\delta(p_1+p_2-p_3-p_4)(q^2+\kappa^2)^{-2}\int d^4k\{\bar{u}(p_3)\gamma_5\tau_i$$

$$\times\ (i\gamma(p_3-k)-\kappa_0)\gamma_5\tau_j(i\gamma(p_1-k)-\kappa_0)\gamma_5\tau_i u(p_1)(k^2+\kappa^2)^{-2}$$

$$\times\ ((k-p_3)^2+\kappa_0^2)^{-1}((k-p_1)^2+\kappa_0^2)^{-1}\bar{u}(p_4)\gamma_5\tau_j u(p_2).$$

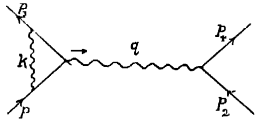

Fig. 1(b)

which simplifies to

$$S^{(b)}_4 = -\frac{f^4\kappa^4\kappa_0^2}{(2\pi)^6(p_{10}p_{20}p_{30}p_{40})^{\frac{1}{2}}}\,\delta(p_1+p_2-p_3-p_4)\bar{u}(p_3)\gamma_5\tau_i u(p_1)\bar{u}(p_4)\gamma_5\tau_i u(p_2)$$

$$\times(q^2+\kappa^2)^{-2}U \quad \ldots \quad (8)$$

wher

$$U = \int \frac{k^2 d^4k}{(k^2-2kp_1)(k^2-2kp_3)(k^2+\kappa^2)^{-2}} \quad (9)$$

The above integral U is convergent. To evaluate it, we use

$$1/(abc^2) = \int_0^1 dx \int_0^1 dy\; 6y(1-y)\{(ax+b(1-x))y+c(1-y)\}^{-4}.$$

and thus obtain, on integrating for the momentum variable,

$$U = i\pi^2 \int_0^1 dx \int_0^1 dy\;\; y(1-y)\;\frac{y^2(\kappa_0^2+q^2x(1-x))+2\kappa^2(1-y)}{(y^2(\kappa_0^2+q^2x(1-x))+\kappa^2(1-y))^2}. \quad \ldots \quad (10)$$

We can carry out charge renormalisation by writing

$$U = i\pi^2\int_0^1 dx \int_0^1 dy\; y(1-y)\{(y^2\kappa_0^2+\kappa^2(1-y))^{-1}+\kappa^2(1-y)(y^2\kappa_0^2+\kappa^2(1-y))^{-2}\}$$

$$+\,i\pi^2 \int_0^1 dx \int_0^1 dy\;\; y(1-y)\{(y^2(\kappa_0^2+q^2x(1-x))+\kappa^2(1-y))^{-1}$$

$$-(y^2\kappa_0^2+\kappa^2(1-y))^{-1}-\kappa^2(1-y)(y^2\kappa_0^2+\kappa^2(1-y))^{-2}$$

$$+\kappa^2(1-y)(y^2(\kappa_0^2+q^2x(1-x))+\kappa^2(1-y))^{-2}\}$$

$$=i\pi^2 \int_0^1 dx \int_0^1 dy\; y(1-y)\frac{y^2\kappa_0^2+2\kappa^2(1-y)}{(y^2\kappa_0^2+\kappa^2(1-y))^2} - i\pi^2 q^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz\; y(1-y)$$

$$\times x(1-x)\{(y^2\kappa_0^2+q^2x(1-x)y^2z+\kappa^2(1-y))^{-2}$$

$$+2\kappa^2(1-y)(y^2\kappa_0^2+q^2x(1-x)y^2z+\kappa^2(1-y))^{-3}\}\,. \quad \ldots \quad (11)$$

In deducing the above result we have utilised the equation

$$\frac{1}{\alpha^n}-\frac{1}{\beta^n}=\int_0^1 \frac{n(\beta-\alpha)}{\{\alpha z+\beta(1-z)\}^{n+1}}\,dz\,. \quad \ldots \quad (12)$$

The first term on the right hand side of Eq. (11) may be regarded as the charge renormalisation term to be included in the second order contribution. This is because we assume that when there is no change in momentum, the contribution from the vertex diagram should vanish. The second term of the same equation gives us the physically important contribution from this process For non-relativistic regions, neglecting q^2 as compared to κ_0^2 and making the substitution $\lambda=\kappa^2/\kappa_0^2$, the second term of Eq. (11) is seen to be

$$\simeq -i\pi^2 q^2\,\frac{1}{6\kappa_0^4}\int_0^1\left\{\frac{y(1-y)dy}{(y^2+\lambda(1-y))^2}+\frac{2\lambda y(1-y)^2dy}{(y^2+\lambda(1-y))^3}\right\}$$

$$\simeq -\frac{i\pi^2q^2}{6\kappa_0^4}\left[\frac{1}{\lambda}-\frac{3}{8\lambda^{\frac{1}{2}}}\cos^{-1}(\tfrac{1}{2}\lambda^{\frac{1}{2}})\right]$$

Thus we have, $U=O\left(\dfrac{q^2}{\kappa_0^2\kappa^2}\right)$, and hence, substituting in Eq. (8) and comparing with the second order contribution, the physical part of the contribution from the vertex diagram is found to be of the order of f^2q^2/κ_0^2 times the second order contribution.

We shall now see the order of magnitude of the renormalisation term. This term, on the right hand side of Eq. (11), is

$$\frac{i\pi^2}{\kappa_0^2}\left[\int_0^1\frac{y^3(1-y)}{(y^2+\lambda(1-y))^2}dy+2\lambda\int_0^1\frac{y(1-y)^2}{(y^2+\lambda(1-y))^2}dy\right]$$

$$\simeq \frac{i\pi^2}{\kappa_0^2}\,[0.8+0.7]=\frac{i\pi^2}{\kappa_0^2}\times 1.5.$$

Thus, by Eqns. (8) and (11), the part of $S^{(b)}{}_4$ that goes as renormalisation term is

$$-\frac{f^4\kappa^4\kappa_0^2}{(2\pi)^6(p_{10}p_{20}p_{30}p_{40})^{\frac{1}{2}}}\;\frac{i\pi^2}{\kappa_0^2}\times 1.5\times\delta(p_1+p_2-p_3-p_4)(q^2+\kappa^2)^{-2}$$

$$\times\bar{u}(p_3)\gamma_5\tau_i u(p_1)\bar{u}(p_4)\gamma_5\tau_i u(p_2).$$

Hence, on comparing with the second order contribution, we obtain,

$$f^2\,(\text{renormalised}) = f^2\left[1 + \frac{f^2\kappa^2}{16\pi^2\kappa_0^2} \times 1.5\right]$$
$$= f^2[1 + f^2 \times 2.14 \times 10^{-4}].$$

This gives us that the renormalisation of the coupling constant is small for any reasonable value of the bare coupling constant, which in this case must be finite. We note that here the bare coupling constant is not an abstraction to be discarded later on, but is a meaningful quantity that can be determined to any degree of accuracy in terms of the physical or renormalised coupling constant.

(c) *Nucleon self-energy* :

The contribution from the nucleon self-energy graph (Fig. 1c) is to be treated just the same way as for ordinary mesons theory, and is to be taken along with the contribution from the diagram Fig. 1c′. We may note that the contribution from both the diagrams gives rise to an ambiguous expression and to get meaningful results we may assume some form of periodic damping explicitly. For example, we may take our interaction Hamiltonian as (Schweber, Bethe and de Hoffman, 1955, p. 286)

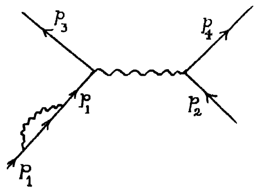

Fig. 1(c)

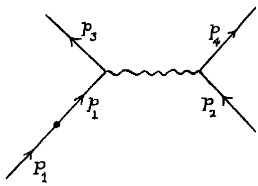

Fig. 1(c′)

$$H_i(x) = if\, g(t)\bar{\Psi}(x)\gamma_5\tau_i\psi(x)\phi^i(x) - \delta\kappa_0(g(t))^2\bar{\Psi}(x)\psi(x) \qquad (13)$$

where

$$g(t) = \int_{-\infty}^{\infty} G(\Gamma_0)e^{-i\Gamma_0 t}\, d\Gamma_0 = \int_{-\infty}^{\infty} G(\Gamma_0)e^{+i\Gamma x}\, d\Gamma_0 \qquad \ldots (14)$$

with $\Gamma = (\Gamma_0, 0, 0, 0)$ and

$$g(0) = \int_{-\infty}^{\infty} G(\Gamma_0)d\Gamma_0 = 1.$$

As an example of a function of such a type, we can take

$$g(t) = (T/t)\sin(t/T) \qquad \ldots (15)$$

such that

$$G(\Gamma_0) = \begin{cases} T/2 & \text{when } -1/T \leqslant \Gamma_0 \leqslant 1/T \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, in the limit T approaches infinity, $G(\Gamma_0)$ approaches a δ-function, and for any finite time, $g(t)$ approaches unity.

Let us now consider the contribution from the self-energy diagram in the general case, with p not the momentum of an external line, given as

$$\Sigma(p) = \frac{3if^2\kappa^2}{(2\pi)^4} \int \frac{i\gamma(p-k)-\kappa_0}{((p-k)^2+\kappa_0^2)(k^2+\kappa^2)^2}\, d^4k$$

$$= A + B(i\gamma p+\kappa_0)+(i\gamma p+\kappa_0)^2\Sigma_c(p), \text{ say.} \qquad \dots \quad (16)$$

Now, when we recalculate the contributions from the Figs 1(c) and 1(c′) with the Hamiltonian given by Eq. (13), by interpretation of the term for mass renormalisation, the constant A in Eq. (16) will exactly cancel with the term before going the limit $T\to\infty$. Also, the constant B in this equation, which at the outset is ambiguous, becomes now well-defined and gives rise to the wave-function renormalisation (also vide Jueh and Rohrlich, 1955, p 185) Again, since in this case we have an external line, the contribution with Σ_c will not be there.

Thus, with such diagrams, the contribution goes only to a mass renormalisation (which is finite and has been calculated by Thirring (1950)) and to a wave-function renormalisation, which also is finite and can be calculated in the standard way), and there is no contribution to the scattering process.

(*d*) *Successive and crossed exchange of two mesons* .

As in the previous cases, the contribution from the Feynman diagram Fig. 1(d) for the case of successive exchange of two mesons becomes, on using the equations with p_1 and p_2 as the four-momenta of free particles,

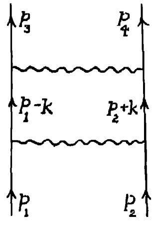

Fig. 1(d)

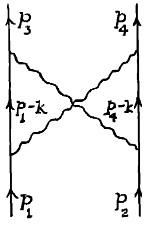

Fig. 1(e)

$$S_4^{(d)} = \frac{f^4\kappa^4\kappa_0^2\ \delta(p_1+p_2-p_3-p_4)}{(2\pi)^4(p_{10}p_{20}p_{30}p_{40})^{\frac{1}{2}}}\ V_{\mu\nu}$$

$$\times\, \bar{u}(p_3)\tau_i\tau_j\gamma_\mu^{(1)}u(p_1)\bar{u}(p_4)\tau_i\tau_j\gamma_\mu^{(2)}u(p_2), \qquad \dots \quad (17)$$

where

$$V_{\mu\nu} = \int d^4k \frac{k_\mu k_\nu}{(k^2+2kp_2)(k^2-2kp_1)((k-q)^2+\kappa^2)^2(k^2+\kappa^2)^2} \quad \ldots \quad (18)$$

Using representation with subsidiary variables and carrying out integration over the momentum variable, we obtain, on simplification,

$$V_{\mu\nu} = 120 \int_0^1 dx \int_0^1 dy \int_0^1 dz \; z(1-z)\, x(1-x)^3 \int d^4k \;\; k_\mu k_\nu [k^2-2k\{p_1 x(1-y)$$

$$-p_2 xy+qz(1-x)\}+q^2 z(1-x)+\kappa^2(1-x)]^{-6}$$

$$= i\pi^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \;\; x(1-x)^3 z(1-z) \left[\frac{6Q_\mu Q_\nu}{D^4} + \frac{g_{\mu\nu}}{D^3} \right], \quad \ldots \quad (19)$$

where

$$Q = p_1 x(1-y) - p_2 xy + qz(1-x) \quad \ldots \quad (20)$$

and

$$D = \kappa_0^2 x^2 (1-2y)^2 + \kappa^2(1-x) + q^2 z(1-z)(1-x)^2 - p^2 x^2 y(1-y). \quad (21)$$

Substituting the value of Q from equation (20), we see that in equation (17),

$$\gamma_\mu^{(1)} \; \gamma_\nu^{(2)} \; Q_\mu \; Q_\nu \simeq -\kappa_0^2 x^2 (1-2y)^2$$

in the nonrelativistic approximation. Also, in this approximation,

$$\gamma_\mu^{(1)} \; \gamma_\nu^{(2)} \; g_{\mu\nu} \simeq 1.$$

Hence we have in Eqns. (17) and (19),

$$\gamma_\mu^{(1)} \gamma_\nu^{(2)} V_{\mu\nu} = i\pi^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \;\; x(1-x)^3 z(1-z) \times \left\{ \frac{-6\kappa_0^2 x^2 (1-2y)^2}{D^4} + \frac{1}{D^3} \right\} \quad (22)$$

where D is given by eqn. (21).

The evaluation of the above integral even in the non-relativistic approximation requires special care, since the denominator vanishes for certain values of the auxiliary variables x and y. The singularity here is to be dealt with by adding a small negative imaginary part to the mass of the meson. The integral to be evaluated is

$$I = \int_0^1 dx \int_0^1 dy \int_0^1 dz \;\; x(1-x)^3 z(1-z) \left\{ -\frac{6\kappa_0^2 x^2 (1-2y)^2}{D^4} + \frac{1}{D^3} \right\},$$

where, for Eq. (21), we can take

$$D \simeq \kappa_0^2 x^2 (1-2y)^2 + \kappa^2(1-x) + q^2 z(1-z)(1-x)^2 - \tfrac{1}{4} p^2 x^2 \quad \ldots \quad (23)$$

We now note that

$$I = \frac{\partial}{\partial q^2} \frac{\partial}{\partial \kappa^2} I_0, \qquad \dots \quad (24)$$

where

$$I_0 = \int_0^1 dz \int_0^1 dx \int_0^1 dy \left\{ -\frac{\kappa_0^2 x^2 (1-2y)^2}{D^2} + \tfrac{1}{2} D^{-1} \right\} \qquad \dots \quad (24')$$

We now put for brevity

$$\phi(x, z) = \kappa^2 (1-x) + q^2 z(1-z)(1-x)^2 - \tfrac{1}{4} p^2 x^2 \qquad \dots \quad (25)$$

and put $s = \kappa_0 x(1-2y)$ as the integration variable instead of y. Then we get,

$$I_0 = \int_0^1 dz \int_0^1 dx \int_0^{\kappa_0 x} ds \, \{-s^2/(s^2 + \phi(x, z))^{-2} + \tfrac{1}{2}(s^2 + \phi(x, z))^{-1}\}. \qquad (26)$$

Since p^2 and q^2 are positive definite constants, (they are positive definite in the centre of mass system), we have, in Eq (25), $\phi(x, z)$ is a monotonically diminishing function of x for any given value of z, and is positive when $x = 0$ and negative when $x = 1$. Hence $\phi(x, z)$ vanishes for one value $\xi = \xi(z)$ of x, with $0 < \xi < 1$, and we have

$$\phi(x, z) > 0 \text{ when } 0 < x < \xi$$

and

$$\phi(x, z) < 0 \quad \text{when } \xi < x < 1.$$

In the second case above, the singularity for the s-integration is to be treated by the addition of a small negative imaginary part. Separating the region of integration of x into $(0, \xi)$ and $(\xi, 1)$ and performing the s-integration for these regions separately, we obtain, after some lengthy but straightforward calculations,

$$I_0 = \frac{1}{2} \int_0^1 dz \int_0^1 dx \frac{x}{[\kappa_0^2 x^2 + \kappa^2 (1-x) + q^2 z(1-z)(1-x)^2 - \frac{1}{4} p^2 x^2]} \cdot \qquad \dots \quad (27)$$

Thus, by Eqns. (24) and (24′),

$$I \simeq \int_0^1 dz \int_0^1 dx \frac{x(1-x)^3 z(1-z)}{[\kappa_0^2 x^2 + \kappa^2 (1-x) + q^2 z(1-z)(1-x)^2]^3}, \qquad \dots \quad (28)$$

where we have again neglected $\frac{1}{4} p^2$ as compared to κ_0^2.

Now, in Eq. (28), we note that when x in $(1-x)$ of the denominator becomes significant, the term $\kappa_0^2 x^2$ becomes large compared with the other terms whereas, for small values of x the terms κ^2 and q^2 predominate in the denominator. Hence in the denominator, we can write 1 instead of $(1-x)$ for the factors of κ^2 and q^2. Again, we can omit x in $(1-x)^3$ in the numerator, since, when x is comparable with unity, the κ_0 in the denominator makes the contribution anyhow small even without this factor, and thus the change thus introduced is negligible. Thus we can write

$$I \simeq \int_0^1 dz \int_0^1 dx \frac{x(1-z)z}{[\kappa_0^2 x^2 + \kappa^2 + q^2 z(1-z)]^3}. \qquad \dots (29)$$

The above heuristic argument, however, does not give us the degree of error in taking the value (29) instead of (28). We can estimate this by taking the difference of the right hand sides of Eqns (28) and (29). Using Eq (12) we can show that this differences is of the order of

$$\int_0^1 \frac{x^2 dx}{[\kappa_0^2 x^2 - Ax + B]^3} + O(\kappa^2) \int_0^1 \frac{x^2 dx}{[\kappa_0^2 x^2 - Ax + B]^4}, \qquad \dots (30)$$

where the quantities A and B do not depend on x and are of the order of κ^2 with $B > A$ Starting with the result

$$J = \int_0^1 \frac{dx}{[\kappa_0^2 x^2 - Ax + B]} = \frac{2}{\sqrt{4\kappa_0^2 B - A^2}} \tan^{-1}\left(\frac{\sqrt{4\kappa_0^2 B - A^2}}{2B - A}\right),$$

we get, by differentiating,

$$\int_0^1 \frac{x^2 dx}{[\kappa_0^2 x^2 - Ax + B]^3} \simeq \frac{1}{16\kappa_0^3 B^{3/2}} \tan^{-1}\left(\frac{2\kappa_0 \sqrt{B}}{2B - A}\right) \qquad \dots (31a)$$

and

$$\int_0^1 \frac{x^2 dx}{[\kappa_0^2 x^2 - Ax + B]^4} \simeq \frac{1}{32\kappa_0^2 B^{5/2}} \tan^{-1}\left(\frac{2\kappa_0^2 \sqrt{B}}{2B - A}\right) \qquad \dots (31b)$$

Thus we find the difference in the approximations (28) and (29) of the integral I as $O\left(\frac{1}{\kappa_0^3 \kappa^3}\right)$. On the other hand we find that the integral (29) is $O(1/\kappa_0^2 \kappa^4)$, such that we have neglected a term of the order of (κ/κ_0) higher than the leading term when we have written down Eq. (29). The small values of the coefficients of the leading terms in Eq. (31) further justify this approximation.

Now performing the x-integration in Eq. (29) and retaining the main contribution, we get,

$$I \simeq \frac{1}{4\kappa_0^2} \int_0^1 \frac{z(1-z)dz}{[\kappa^2+q^2z(1-z)]^2}, \qquad \dots (32)$$

which on direct evaluation gives us

$$I \simeq \frac{1}{4\kappa_0^2} \left\{ \frac{2\kappa^2+q^2}{4(q^2)^{3/2}(\kappa^2+\frac{1}{4}q^2)^{3/2}} \log \left(\frac{\sqrt{\kappa^2+\frac{1}{4}q^2}+\frac{1}{2}(q^2)^{1/2}}{\sqrt{\kappa^2+\frac{1}{4}q^2}-\frac{1}{2}(q^2)^{1/2}} \right) - \frac{1}{2q^2(\kappa^2+\frac{1}{4}q^2)} \right\}, \qquad \dots (33)$$

such that by Eqns. (17) and (29),

$$S_4^{(d)} = \frac{i\pi^2 f^4\kappa^4\kappa_0^2}{(2\pi)^6(p_{10}p_{20}p_{30}p_{40})^{1/2}} \delta(p_1+p_2-p_3-p_4)I \times \bar{u}(p_3)\tau_i\tau_j u(p_1)\bar{u}(p_4)\tau_i\tau_j u(p_2). \qquad \dots (34)$$

For crossed exchange of two mesons, we have, from Fig. 1(e),

$$S_4^{(e)} = -f^4\kappa^4(2\pi)^{-6}\kappa_0^2(p_{10}p_{20}p_{30}p_{40})^{-\frac{1}{2}}\delta(p_1+p_2-p_3-p_4)V'_{\mu\nu}$$
$$\times \bar{u}(p_3)\tau_i\tau_j\gamma_\mu^{(1)}u(p_1)\bar{u}(p_4)\tau_j\tau_i\gamma_\nu^{(2)}u(p_2), \qquad \dots (35)$$

where

$$V'_{\mu\nu} = \int d^4k \frac{k_\mu k_\nu}{(k^2-2kp_4)(k^2-2kp_1)(k^2-2kq+q^2+\kappa^2)^2(k^2+\kappa^2)^2} \qquad \dots (36)$$

Proceeding as in the case of successive exchange of two mesons, we obtain,

$$V'_{\mu\nu} = i\pi^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz\ x(1-x)^3z(1-z) \left\{\frac{6Q'_\mu Q'_\nu}{D'^4} + \frac{g_{\mu\nu}}{D'^3}\right\} \qquad \dots (37)$$

where

$$Q'_\mu = p_{1\mu}x(1-y)+p_{4\mu}xy+q_\mu z(1-x) \qquad \dots (38)$$

and

$$D' = \kappa_0^2x^2+\kappa^2(1-x)+p^2x^2y(1-y)+q^2z(1-z)(1-x)^2+q^2x^2y(1-y)$$
$$\simeq \kappa_0^2x^2+\kappa^2(1-x)+q^2z(1-z)(1-x)^2. \qquad \dots (39)$$

In the further nonrelativistic approximation, we have, in Eq. (35),

$$\gamma_\mu^{(1)}\gamma_\nu^{(2)}V'_{\mu\nu} = i\pi^2 \int_0^1 dx \int_0^1 dz \;\; x(1-x)^3 z(1-z) \left\{-\frac{6\kappa_0^2x^2}{D'^4}+\frac{1}{D'^3}\right\}$$

$$= i\pi^2 I' \quad \text{say.} \qquad \dots \quad (40)$$

We note that D' is positive definite and hence the integration above does not have the complications of the previous case.

For the evaluation of the integral (40), we make the same approximation as was made in the previous case in going from Eqn. (28) to (29). We have seen that this is strictly valid upto an order (κ/κ_0). With this approximation we have, on carrying out the x-integration,

$$I' = -\frac{1}{4\kappa_0^2}\int_0^1 \frac{z(1-z)dz}{[\kappa^2+q^2z(1-z)]^2}\,. \qquad \dots \quad (41)$$

A comparison of the above Eq. with Eq. (32) shows that

$$I' = -I.$$

Hence, by Eqns. (34), (35) and (49) and (41), we obtain, with I given by Eq. (33).

$$S_4^{(d)} + S_4^{(e)} \simeq 6i\pi^2 f^4\kappa^4(2\pi)^{-6}\delta(p_1+p_2-p_3-p_4)I\bar{u}(p_3)u(p_1)\bar{u}(p_4)u(p_2).$$

A comparison with the other terms of the fourth order also shows us that the major contribution in the fourth order arises from the successive and crossed exchange of two mesons. This can also be seen from physical reasons For γ_5 coupling, a transition from particle to antiparticle states will give us the maximum contribution, which is possible for the above type of diagrams.

APPENDIX

Meson mass renormalisation In order to interpret certain terms arising from Eq (4) as mass renormalisation, we have to first write the integral (5) occuring there in the form (6). We notice that this integral, written as

$$I = \int_0^1 dx \int d^4k \; \frac{k^2+qk+\kappa_0^2}{(k^2+2kqx+q^2x+\kappa_0^2)^2} \qquad \dots \quad (A1)$$

is quadratically divergent, and thus requires care even in the change of momentum variables to be integrated. Using well-known methods (Jauch and Rohrlich 1955, also Eq (12)), we obtain, for a shift of the origin,

$$I = \int_0^1 dx \int d^4k \;\; \frac{k^2-q^2x(1-x)+\kappa_0^2}{(k^2+q^2x(1-x)+\kappa_0^2)^2} - \frac{i\pi^2}{4}q^2 \qquad (A2)$$

In order to write the integral I of (A2) in the form (6), we repeatedly make use of the identity (Eq. (12))

$$\frac{1}{\alpha^n}-\frac{1}{\beta^n}=\int_0^1 \frac{n(\beta-\alpha)}{(\alpha z+\beta(1-z))^{n+1}}\,dz$$

This gives us, after some lengthy calculation and comparing with Eq. (6), the values of the constants A and B as

$$A=\int_0^1 dx\int d^4k\,\frac{k^2+\kappa^2x(1-x)+\kappa_0^2}{(k^2-\kappa^2x(1-x)+\kappa_0^2)^2}+\frac{i\pi^2}{4}\kappa^2 \qquad \ldots \quad (A3)$$

and

$$B=\int_0^1 dx\int d^4k\left\{\frac{4x(1-x)(k^2+\kappa_0^2)}{(k^2-\kappa^2x(1-x)+\kappa_0^2)^3}+\frac{x(1-x)}{(k^2-\kappa^2x(1-x)+\kappa_0^2)^2}\right\}-\frac{i\pi^2}{4}\cdot \quad (A4)$$

Also, after momentum integration, the finite integral $I_c(q^2)$ is given as

$$I_c(q^2)=i\pi^2\int_0^1 dx\int_0^1 dy\int_0^1 dz\; x^2(1-x)^2y\times\{3(\kappa_0^2+q^2x(1-x)yz-\kappa^2x(1-x)(1-yz))^{-1}$$

$$+2\kappa_0^2(\kappa_0^2+q^2x(1-x)yz-\kappa^2x(1-x)(1-yz))^{-2}\}. \qquad \ldots \quad (A5)$$

We shall now see how the divergent constants A and B may be interpreted as mass renormalisation terms. For this purpose, we write the meson field Lagrangian and the interaction Lagrangizn in the Heisenberg representation as

$$\underline{L}_m+\underline{L}_i$$

$$=-\tfrac{1}{2}\,[(\Box\underline{\phi}^i)(\Box\underline{\phi}^i)+(\kappa_1^2+\delta\kappa_1^2+\kappa_2^2+\delta\kappa_2^2)(\partial_\mu\underline{\phi}^i)(\partial_\mu\underline{\phi}^i)$$

$$+(\kappa_1^2+\delta\kappa_1^2)(\kappa_2^2+\delta\kappa_2^2)\underline{\phi}^i\underline{\phi}^i]-if\underline{\bar{\Psi}}\gamma_5\tau_i\underline{\psi}\,\underline{\phi}^i$$

$$+\frac{1}{2\kappa^2}[(\delta\kappa_1^2+\delta\kappa_2^2)(\partial_\mu\underline{\phi}^i)(\partial_\mu\underline{\phi}^i)+(\kappa_1^2\delta\kappa_2^2+\kappa_2^2\delta\kappa_1^2+\delta\kappa_1^2\delta\kappa_2^2)\underline{\phi}^i\underline{\phi}^i], \quad \ldots \quad (A6)$$

where the line below the operators means the corresponding quantities in Heisenberg representation. In the above, κ_1 and κ_2 are unrenormalised masses and $\delta\kappa_1^2$ and $\delta\kappa_2^2$ are the renormalisation terms, such that the observed mass is given the equations

$$\kappa_1^2+\delta\kappa_1^2+\kappa_2^2+\delta\kappa_2^2=2\kappa^2,$$

$$(\kappa_1^2+\delta\kappa_1^2)(\kappa_2^2+\delta\kappa_2^2)=\kappa^4 \qquad (A7)$$

We now have to set up the interaction representation and to do this, we proceed as in Umezawa (1956).

Using Eq. ($A7$), the meson field equation becomes

$$(\Box-\kappa^2)^2\underline{\phi}^i(x)=-if\kappa^2\underline{\bar{\psi}}(x)\gamma_5\tau_i\underline{\psi}(x)+(\kappa_1^2\delta\kappa_2^2+\kappa_2^2\delta+\kappa_1^2\delta\kappa_2^2)\underline{\phi}^i(x)$$
$$-(\delta\kappa_1^2+\delta\kappa_2^2)\Box\underline{\phi}^i(x) \qquad \text{... (A8)}$$

We now solve the above equation with the retarded Green's function $G_R(x-x')$ satisfying the equation

$$(\Box-\kappa^2)^2G_R(x-x')=-\delta_4(x-x')$$

and $G_R(x-x')=0$ when $x_0<x'_0$ and the function $G(x)=G_R(x)-G_R(-x)$. Then, a partial integration with the assumption that the interaction (and hence the renormalisation terms) vanish at infinite past sufficiently rapidly, gives us,

$$\underline{\phi}^i(x)=\phi^i(x)+\tfrac{1}{2}\int_{-\infty}^{\infty}d^4x'(1+\epsilon(x-x'))\{G(x-x')[if\kappa^2\underline{\bar{\psi}}(x')\gamma_5\tau_i\underline{\psi}(x')$$
$$-(\kappa_1^2\delta\kappa_2^2+\kappa_2^2\delta\kappa_1^2+\delta\kappa_1^2\delta\kappa_2^2)\underline{\phi}_i(x')]-(\delta\kappa_1^2+\delta\kappa_2^2)(\partial'_\mu G(x-x'))(\partial'_\mu\underline{\phi}^i(x'))\} \quad \text{(A9)}$$

where $\phi^i(x)$ is a field operator satisfying the equation

$$(\Box-\kappa^2)^2\phi^i(x)=0 \qquad \text{... (A10)}$$

and $\epsilon(x-x')=1$ or -1 according as x' is earlier or later than x. The words "earlier" and "later" refer to space-like surfaces on which $\epsilon(x-x')$ has a discontinuity.

Now, let the interaction representation state-vector $\underline{\psi}(\sigma)$ be given in terms of the Heisenberg state-vector Φ as $\Phi(\sigma)=S(\sigma)\Phi$, $S(-\infty)=1$. Then we have,

$$i\,\frac{\delta S(\sigma)}{\delta\sigma(x)}=H(x,\sigma)\,S(\sigma) \qquad \text{(A11)}$$

and the integrability condition

$$\frac{\delta H(x',\sigma)}{\delta\sigma(x)}-\frac{\delta H(x,\sigma)}{\delta\sigma(x')}=i[H(x',\sigma),H(x,\sigma)] \qquad \text{(A12)}$$

where x and x' are any two points lying on the surface σ.

We now define auxiliary field operators $\phi^i(x,\sigma)$ given by

$$\phi^i(x,\sigma)=\phi^i(x)+\int_{-\infty}^{\sigma}\{G(x-x')[if\kappa^2\underline{\bar{\psi}}(x')\gamma_5\tau_i\psi(x')$$
$$-(\kappa_1^2\delta\kappa_2^2+\kappa_2^2\delta\kappa_2^2)+\kappa_1^2\kappa_2^2)\underline{\phi}^i(x')]$$
$$-(\delta\kappa_1^2+\delta\kappa_2^2)(\partial'_\mu G(x-x'))(\partial_\mu\phi^i(x'))\}d^4x', \qquad \text{(A13)}$$

and proceed exactly as in Umezawa (1956). Then we obtain that the interaction Hamiltonian is given by the equation

$$[\phi\,(x), H(x', \sigma)] = iG(x-x')S(\sigma)[if\,\kappa^2\bar{\Psi}(x')\gamma_5\tau_i\psi(x')$$
$$-(\kappa_1{}^2\delta\kappa_2{}^2+\kappa_2{}^2\delta\kappa_1{}^2+\delta\kappa_1{}^2\delta\kappa_2{}^2)\phi^i(x')]S^{-1}(\sigma)$$
$$-S(\sigma)(\delta_1{}^2+\delta_2{}^2)(\partial'_\mu G(x-x'))(\partial'_\mu\phi^i(x'))S^{-1}(\sigma), \qquad \ldots \quad \text{(A14)}$$

where clearly x' lies on σ. Using this, and that $G(x-x')$ and its first order space-time derivatives vanish for space-like separation of the points, Eq. (A14) may be seen to reduce to

$$[\phi^i(x), H(x', \sigma)] = i\kappa^2G(x-x')if\,\bar{\phi}(x')\gamma_5\tau_i\psi(x')$$
$$-(\kappa_1{}^2\delta\kappa_2{}^2+\kappa_2{}^2\delta\kappa_1{}^2+\delta\kappa_1{}^2\delta\kappa_2{}^2)iG(x-x')\phi^i(x')$$
$$-(\delta\kappa_1{}^2+\delta\kappa_2{}^2)i(\partial'_\mu G(x-x'))(\partial'_\mu\phi^i(x')) \qquad \ldots \quad \text{(A15)}$$

Since the field operators in Eq (A15) satisfy free field commutation relations

$$[\phi^i(x), \phi^j(x')] = i\kappa^2\delta_{ij}G(x-x'),$$

we obtain

$$H(x', \sigma) = if\bar{\Psi}(x')\gamma_5\tau_i\psi(x')\phi^i(x) - \tfrac{1}{2}\kappa^2(\kappa_1{}^2\delta\kappa_2{}^2+\kappa_2{}^2\delta\kappa_1{}^2+\delta\kappa_1{}^2\delta\kappa_2{}^2)\phi^i(x')\phi^i(x')$$
$$-\tfrac{1}{2}\kappa^2(\delta\kappa_1{}^2+\delta\kappa_2{}^2)(\partial'_\mu\phi^i(x'))(\partial'_\mu\phi^i(x')) \qquad \ldots \quad \text{(A16)}$$

We note that the presence of derivatives of field operations in the self-coupling of the meson fields *does not* introduce terms depending on the normal to the surface

The above interaction Hamiltonian clearly satisfies the integrability condition ($A12$).

The presence of second and third terms in $H(x, \sigma)$ of Eq. ($A16$) gives rise to vertices with only two meson lines. Consideration of the corresponding Feyman diagram (Fig. $1f'$) gives us, with

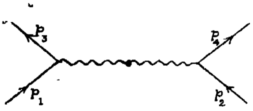

Fig. $1(f')$

$$A_1 = \tfrac{1}{2}\kappa^2(\kappa_1{}^2\,\delta\kappa_2{}^2+\kappa_2{}^2\delta\kappa_1{}^2+\delta\kappa_1{}^2\delta\kappa_2{}^2)$$
$$B_1 = \tfrac{1}{2}\kappa^2(\delta\kappa_1{}^2+\delta\kappa_2{}^2) \qquad \ldots \quad \text{(A17)}$$

the contribution as

$$S_2{}^{(a')} = -if^2\kappa^4(2\pi)^{-2}\delta(p_1+p_2-p_3-p_4)\kappa_0{}^2(p_{10}p_{20}p_{30}p_{40})^{-\frac{1}{2}}$$
$$\times\bar{u}(p_3)\gamma_5\tau_i u(p_1)\bar{u}(p_4)\gamma_5\tau_i u(p_2)[(A_1-B_1\kappa^2)(q^2+\kappa^2)^{-4}+B_1(q^2+\kappa^2)^{-3}]. \quad \ldots \quad \text{(A18)}$$

Comparison of Eqns. ($A17$), (4) and (6) shows that the contributions with the divergent constants A and B are cancelled provided

$$4f^4\kappa^4 A = if^2\kappa^4(2\pi)^4(A_1 - B_1\kappa^2),$$

$$4f^4\kappa^4 B = if^2\kappa^4(2\pi)^4 B_1.$$

Remembering substitution (17), the last equation gives the bare masses and the renormalisation terms with A and B given by equations (A3) and (A4). We note note that the interpretation of the constants A and B in conventional meson theory went to both mass and coupling constant renormalisation, which is not possible here. Also, two bare masses were needed to interpret two divergent constants as renormalisation terms. On the other hand, if we had started with a bare particle of unique mass, and had considered interaction, the corresponding mass renormalisation would have given rise to a splitting of the mass. This result is interesting, but such a splitting is useless to investigate so long as the equations that determine the splitting, and thus the physical masses, contain infinite constants.

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